# Infinite-dimensional Representations of Algebras

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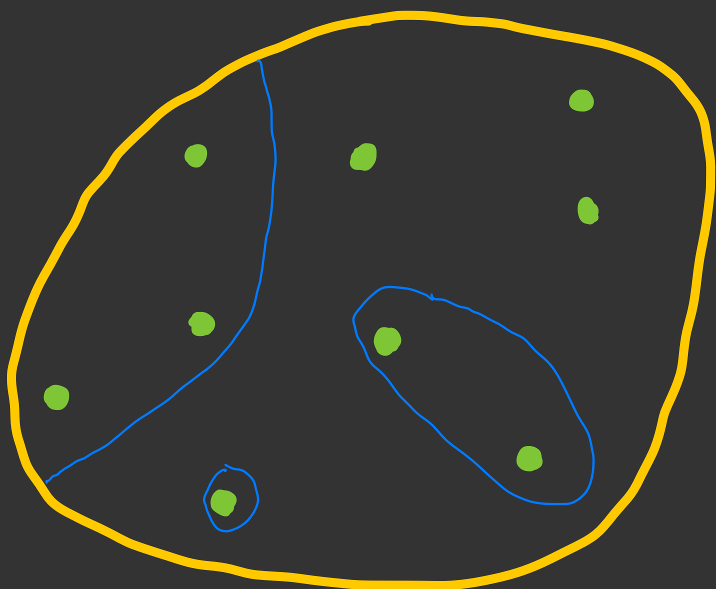
# Plan for the lectures

K : field

A : algebra over K

M : A-module (K-vector space with an A-action)



A-Mod : (category of) all A-modules

**The Ziegler Spectrum:**

**Points:** (iso-classes of) Indecomposable pure-injectives

**Topology:** (determined by) Finite matrix subgroups

## Plan for the lectures

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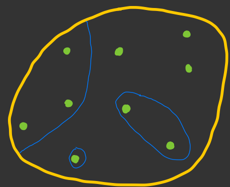


A-Mod : (category of) all A-modules

**The Ziegler Spectrum:**

Points: (iso-classes of) 3
Indecomposable pure-injectives

Topology: (determined by)
Finite-matrix subgroups 2

**Part 1:** Examples of K-algebras and modules:

1. field,
2. polynomial ring,
3. path algebra.

**Part 2:** Finite-matrix subgroups and pp-definable subgroups of a module. Pure-injective modules and examples.

**Part 3:** The Ziegler spectrum and finite representation type of an algebra.

# K-algebras and their modules

Example: The field K

$$\boxed{\text{K-module} = \text{K-vector space}}$$

Every K-vector space M has a basis $\rightsquigarrow$ M is isomorphic to $K^{(\alpha)}$

Direct sum of copies of K indexed by $\alpha$

# Example: The polynomial ring K[X]

## What is a K[X]-module?

$$V \circlearrowleft K[X]$$

K-vector space
with a K[X]-action

$$P := k_0 + k_1 X + k_2 X^2 + \ldots + k_n X^n$$

$$\begin{aligned} P.v &= (k_0 + k_1 X + \ldots + k_n X^n).v \\ &= k_0 v + k_1 (X.v) + \ldots + k_n (X^n.v) \end{aligned}$$

K[X]-action depends on
K-vectorspace and action of X

# What is a K[X]-module?

$$V \circlearrowleft \Phi$$

**K-vector space with a K-linear endomorphism**

Example: Fix $m \geq 0$, $k \in K$

$$J_{k,m} : K^m \longrightarrow K^m$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \longmapsto \begin{pmatrix} k & 1 & 0 & 0 & \dots & 0 \\ 0 & k & 1 & 0 & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & k \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$x_1, \dots, x_m \in K$

Jordan block

**Exercise:** Show this defines the module $K[X]/(X-k)^m$

$$k[x]/(x-k) \longrightarrow k[x]/(x-k)^2 \longrightarrow k[x]/(x-k)^3 \longrightarrow k[x]/(x-k)^4 \longrightarrow \cdots$$

$$\begin{tikzcd}[column sep=large]
K \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[loop below, "k"] & K^2 \arrow[r, "{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}}"] \arrow[loop below, "{\begin{bmatrix}k&1\\0&k\end{bmatrix}}"] & K^3 \arrow[r, "{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}}"] \arrow[loop below, "{\begin{bmatrix}k&1&0\\0&k&1\\0&0&k\end{bmatrix}}"] & K^4 \arrow[r] \arrow[loop below, "{\begin{bmatrix}k&1&0&0\\0&k&1&0\\0&0&k&1\\0&0&0&k\end{bmatrix}}"] & \cdots
\end{tikzcd}$$

$$K[X]/(X-k) \longrightarrow K[X]/(X-k)^2 \longrightarrow K[X]/(X-k)^3 \longrightarrow K[X]/(X-k)^4 \longrightarrow \cdots \quad \varinjlim_{m \in \mathbb{N}} K[X]/(X-k)^m$$

The direct limit of the sequence:

- Colimit in the category of K[X]-modules
- There is an explicit construction

**The k-Prüfer module**

$$K^{(\mathbb{N})} \circlearrowleft \begin{bmatrix} k & 1 & 0 & 0 & 0 & \cdots \\ 0 & k & 1 & 0 & 0 & \cdots \\ 0 & 0 & k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**Example:** The path algebra $\Lambda = KQ/\langle \rho \rangle$ where

$$Q := \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[rr, "\beta"] & & 2 \arrow[loop right, "\delta"] \end{tikzcd} \qquad \rho := \{ \alpha^2 , \delta^2 , \delta\beta\alpha \}$$

$\cong$ K-vector space with basis $\{ e_1, e_2, \alpha, \beta, \delta, \beta\alpha, \delta\beta \}$
with multiplication given by concatenation of paths

**What is a $\Lambda$-module?**

$$\begin{tikzcd} V_1 \arrow[loop left, "V_\alpha"] \arrow[rr, "V_\beta"'] & & V_2 \arrow[loop right, "V_\delta"] \end{tikzcd}$$

Two K-vector spaces and three K-linear maps
such that $V_\alpha^2 = 0$ , $V_\delta^2 = 0$ , $V_\delta V_\beta V_\alpha = 0$

$$Q := \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[rr, "\beta"] & & 2 \arrow[loop right, "\delta"] \end{tikzcd} \qquad \rho := \{\alpha^2, \delta^2, \delta\beta\alpha\}$$

For every “string”, we can define a module, as follows:

$$Z_1 = \begin{tikzcd}[column sep=small] v_1 \arrow[dr, "\alpha"] & & & & v_3 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \\ & v_2 \arrow[dr, "\beta"] & & w_2 \arrow[dl, "\delta"'] & & v_4 \arrow[dr, "\beta"] & \\ & & w_1 & & & & w_3 \end{tikzcd}$$

$$M(Z_1) = \begin{tikzcd} K^4 \arrow[loop left, "{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}"] \arrow[rr, "{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}"] & & K^3 \arrow[loop right, "{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}"] \end{tikzcd}$$

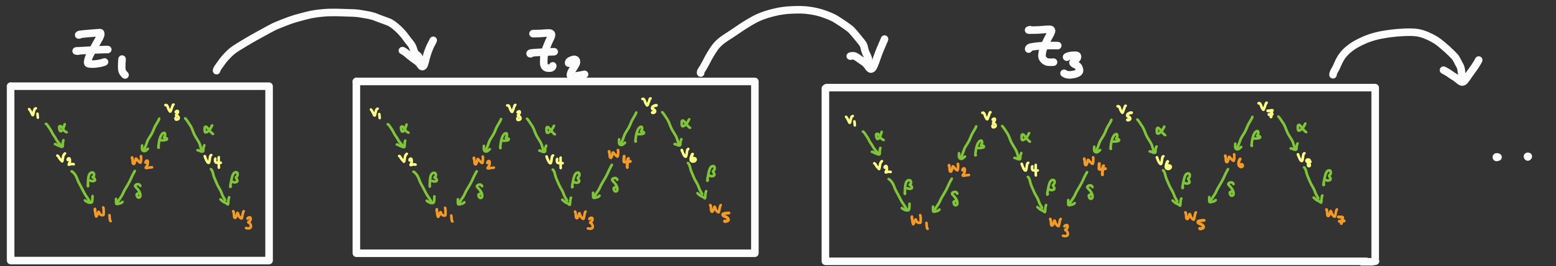
# Another example:

$$Z_2 = \begin{tikzcd}
v_1 \arrow[dr, "\alpha"] & & & & v_3 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & & & v_5 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \\
& v_2 \arrow[dr, "\beta"] & & w_2 \arrow[dl, "\delta"'] & & v_4 \arrow[dr, "\beta"] & & w_4 \arrow[dl, "\delta"'] & & v_6 \arrow[dr, "\beta"] & \\
& & w_1 & & & & w_3 & & & & w_5
\end{tikzcd}$$

$$M(Z_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\qquad
\begin{tikzcd} K^6 \arrow[loop left] \arrow[r] & K^5 \arrow[loop right] \end{tikzcd}
\qquad
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Once again we have a sequence of homomorphism:




$$
\begin{tikzcd}
K^4 \arrow[loop above] \arrow[r, "\begin{bmatrix} id \\ 0 \\ 0 \end{bmatrix}"] \arrow[d] & K^6 \arrow[loop above] \arrow[r, "\begin{bmatrix} id \\ 0 \\ 0 \end{bmatrix}"] \arrow[d] & K^8 \arrow[loop above] \arrow[r, "\begin{bmatrix} id \\ 0 \\ 0 \end{bmatrix}"] \arrow[d] & \cdots \\
K^3 \arrow[loop below] \arrow[r, "\begin{bmatrix} id \\ 0 \\ 0 \end{bmatrix}"] & K^5 \arrow[loop below] \arrow[r, "\begin{bmatrix} id \\ 0 \\ 0 \end{bmatrix}"] & K^7 \arrow[loop below] \arrow[r, "\begin{bmatrix} id \\ 0 \\ 0 \end{bmatrix}"] & \cdots
\end{tikzcd}
$$

$M(Z_1)$ $\qquad$ $M(Z_2)$ $\qquad$ $M(Z_3)$

# The direct limit yields a $\Lambda$-module:

$$Z = \begin{tikzcd}[column sep=small]
v_1 \arrow[dr,"\alpha"] & & & & v_3 \arrow[dl,"\beta"'] \arrow[dr,"\alpha"] & & & & v_5 \arrow[dl,"\beta"'] \arrow[dr,"\alpha"] & & & & v_7 \arrow[dl,"\beta"'] \arrow[dr,"\alpha"] & & & & \\
& v_2 \arrow[dr,"\beta"] & & w_2 \arrow[dl,"\delta"'] & & v_4 \arrow[dr,"\beta"] & & w_4 \arrow[dl,"\delta"'] & & v_6 \arrow[dr,"\beta"] & & w_6 \arrow[dl,"\delta"'] & & v_8 \arrow[dr,"\beta"] & & w_8 \arrow[dl,"\delta"'] & \cdots\cdots \\
& & w_1 & & & & w_3 & & & & w_5 & & & & w_7 & &
\end{tikzcd}$$

$$M(Z) = \boxed{\begin{tikzcd}
K^{(\mathbb{N})} \arrow[loop left, "{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \\ & \vdots & & \vdots & & \vdots & \ddots \end{bmatrix}}"] \arrow[r, "{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & & \vdots & \vdots & & \ddots \end{bmatrix}}"'] & K^{(\mathbb{N})} \arrow[loop right, "{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}}"]
\end{tikzcd}}$$

# K-duality

Recall from linear algebra:
Given a K-vector space V, we define the dual K-vector space $V^* = \mathrm{Hom}_K(V, K)$.

This defines a functor $\mathrm{Hom}_K(-, K) : \text{Vect-}K \longrightarrow \text{Vect-}K$.

**Definition:** Let A be a K-algebra.
Given an A-module M, we define the K-dual $A^{op}$-module $M^* = \mathrm{Hom}_K(M, K)$ with $A^{op}$-action $A^{op} \times M^* \longrightarrow M^*$ given by

$$(a, f) \longmapsto af \quad \text{where } (af)(m) = f(am).$$

**Remarks:**

1. If $A$ is commutative, then $A = A^{op}$ e.g. $K$ or $K[X]$.
2. $(A^{op})^{op} = A$
3. There is an equivalence of categories
$$A^{op}\text{-Mod} \cong \text{Mod-}A$$
4. Duality defines a functor
$$\mathrm{Hom}_K(-, K) : A\text{-Mod} \longrightarrow A^{op}\text{-Mod}$$

**Examples:**

Direct product of copies of $K$ indexed by $\alpha$.

1. The $K$-dual of $K^{(\alpha)}$ is $K^{\alpha}$.
2. The $K$-dual of the $k$-Prufer module is called the $k$-adic module.

**Exercise**: Show that $\Lambda^{op} = KQ^*/ <\rho^*>$ where

$$Q^* := \begin{tikzcd} 1 \arrow[loop left, "\alpha^*"] & & 2 \arrow[ll, "\beta^*"'] \arrow[loop right, "\delta^*"] \end{tikzcd} \qquad \rho^* := \{(\alpha^*)^2, (\delta^*)^2, \alpha^*\beta^*\delta^*\}$$

**What is a $\Lambda^{op}$-module?**

$$\begin{tikzcd} V_1 \arrow[loop left, "V_{\alpha^*}"] & & V_2 \arrow[ll, "V_{\beta^*}"'] \arrow[loop right, "V_{\delta^*}"] \end{tikzcd}$$

Two K-vector spaces and three K-linear maps such that

$$V_{\alpha^*}^2 = 0, \quad V_{\delta^*}^2 = 0, \quad V_{\alpha^*}V_{\beta^*}V_{\delta^*} = 0$$

$$Q^* := \begin{tikzcd} 1 \arrow[loop left, "\alpha^*"] & 2 \arrow[l, "\beta^*"'] \arrow[loop right, "\delta^*"] \end{tikzcd} \qquad \rho^* := \{(\alpha^*)^2,\ (\delta^*)^2,\ \alpha^*\beta^*\delta^*\}$$

Consider the string

$$y^* = w_1 \xrightarrow{\delta^*} w_2 \xrightarrow{\beta^*} v_1 \xleftarrow{\alpha^*} v_2 \xleftarrow{\beta^*} w_3 \xrightarrow{\delta^*} w_4 \xrightarrow{\beta^*} v_3 \xleftarrow{\alpha^*} v_4 \xleftarrow{\beta^*} w_5 \xrightarrow{\delta^*} w_6 \xrightarrow{\beta^*} v_5 \xleftarrow{\alpha^*} v_6 \xleftarrow{\beta^*} w_7 \xrightarrow{\delta^*} w_8 \xrightarrow{\beta^*} v_7 \xleftarrow{\alpha^*} v_8 \cdots\cdots$$

The K-dual M(y*)* is

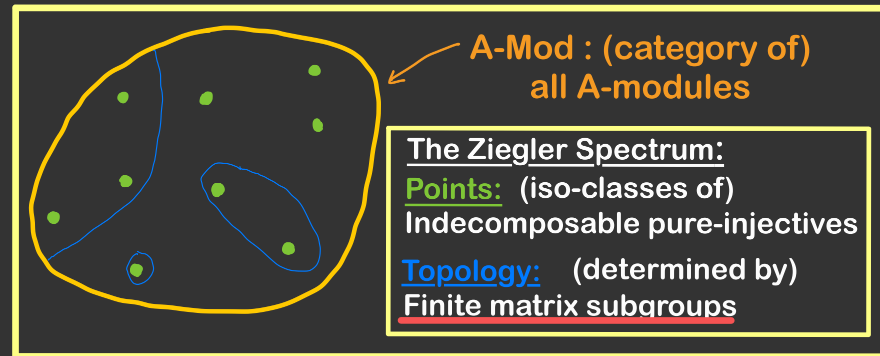
$$\begin{tikzcd} K^{\mathbb{N}} \arrow[loop left] \arrow[r] & K^{\mathbb{N}} \arrow[loop right] \end{tikzcd}$$

with the action:

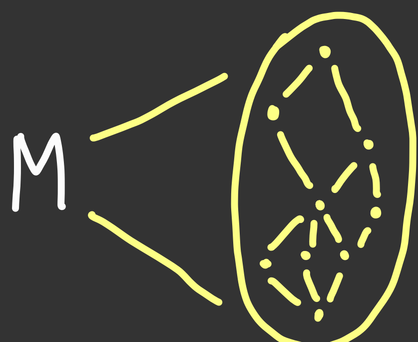
$$w_1 \xleftarrow{\delta} w_2 \xleftarrow{\beta} v_1 \xrightarrow{\alpha} v_2 \xrightarrow{\beta} w_3 \xleftarrow{\delta} w_4 \xleftarrow{\beta} v_3 \xrightarrow{\alpha} v_4 \xrightarrow{\beta} w_5 \xleftarrow{\delta} w_6 \xleftarrow{\beta} v_5 \xrightarrow{\alpha} v_6 \xrightarrow{\beta} w_7 \xleftarrow{\delta} w_8 \xleftarrow{\beta} v_7 \xrightarrow{\alpha} v_8 \xrightarrow{\beta} w_9 \xleftarrow{\delta} w_{10} \cdots\cdots$$

# Finite matrix subgroups

Recall that we aim to define the Ziegler Spectrum of A




## Finite matrix subgroups of an A-module M:




- distinguished K-subspaces of M
- form a lattice
- each one determined by a pointed finitely presented module (N, n)

**Definition:** Let M be an A-module. A K-subspace H of M is called a **finite matrix subgroup** if there exists a finitely presented A-module N and an element n of N and

$$H = \{ f(n) \mid f: N \to M \}.$$

Finitely presented means that there exists an exact sequence

$$A^m \to A^n \to N \to 0$$

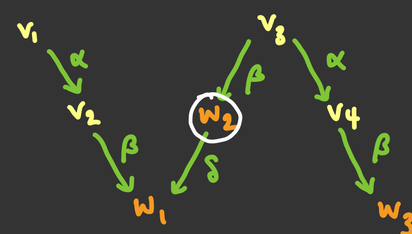
**Notation:** Let (N,n) be a pointed finitely presented module. Define

$$H_{N,n}(M) = \{ f(n) \mid f: N \to M \}.$$

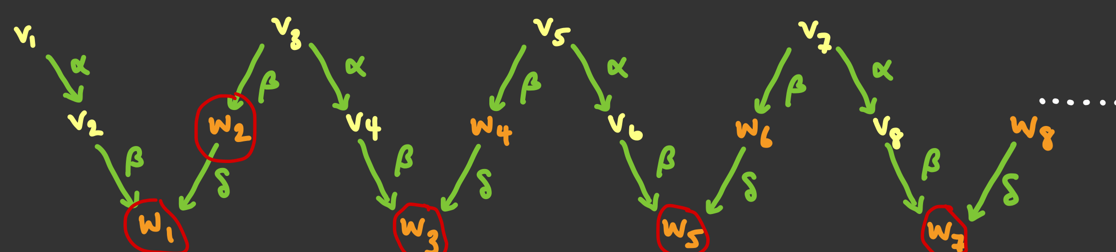
**Example:** Consider the path algebra $\Lambda = KQ/\langle \rho \rangle$ where

$$Q := \overset{\alpha}{\circlearrowleft} 1 \xrightarrow{\beta} 2 \circlearrowright^{\delta} \qquad \rho := \{\alpha^2, \delta^2, \delta\beta\alpha\}$$

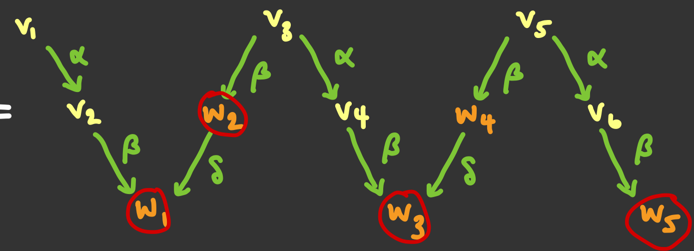
and the pointed finitely presented module $(M(z_1), w_2)$:




The finite matrix subgroup $H_{M(z_1),w_2}(M(z))$:



**Example:** The following diagram indicates the finite matrix subgroup $H_{M(z_1), w_2}(M(z_2))$:

$z_2 =$ 


**Remark:** Let $(N,n)$ be a pointed finitely presented A-module.

There is a functor $H_{N,n}$ : Mod-A $\longrightarrow$ Vect-K such that an A-module M is sent to the vector space $H_{N,n}(M)$.

# Pp-definable subgroups

Let M be an A-module. A set of A-linear equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$$

determines a K-subspace of M:

$$\{ m_1 \in M \mid \exists m_2, \ldots, m_n : a_{j1}m_1 + \ldots + a_{jn}m_n = 0 \ \forall 1 \leq j \leq m \}$$

Note that the equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

are determined by the (m x n)-matrix $P := (a_{ji})$.

$$\phi_P(M) := \{m_1 \in M \mid \exists m_2, \dots, m_n : a_{j1}m_1 + \dots + a_{jn}m_n = 0 \ \forall 1 \leq j \leq m\}$$

<u>Definition:</u> A K-subspace of the form $\phi_P(M)$ for some (m x n)-matrix P is called a **pp-definable subgroup** of M.

**Example:** Consider the path algebra $\Lambda$. Recall that

$$Q := \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\delta"] \end{tikzcd} \qquad \rho := \{\alpha^2, \delta^2, \delta\beta\alpha\}$$

and consider the system of $\Lambda$-linear equations:

$$\left.\begin{aligned} e_1 x_1 &= 0 \\ \delta x_1 - x_2 &= 0 \\ \beta x_3 - x_2 &= 0 \\ \alpha x_4 - x_3 &= 0 \\ \beta x_4 &= 0 \end{aligned}\right\} \Rightarrow \quad P = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ 0 & -1 & \beta & 0 \\ 0 & 0 & -1 & \alpha \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

$$\left.\begin{aligned} e_1 x_1 &= 0 \\ \delta x_1 - x_2 &= 0 \\ \beta x_3 - x_2 &= 0 \\ \alpha x_4 - x_3 &= 0 \\ \beta x_4 &= 0 \end{aligned}\right\} \Rightarrow \quad P = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ 0 & -1 & \beta & 0 \\ 0 & 0 & -1 & \alpha \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

**Consider the module M(z) where**

$$Z = \begin{tikzcd}
v_1 \arrow[dr, "\alpha"] & & & & v_3 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & & & v_5 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & & & v_7 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & & & \\
& v_2 \arrow[dr, "\beta"] & & w_2 \arrow[dl, "\delta"'] & & v_4 \arrow[dr, "\beta"] & & w_4 \arrow[dl, "\delta"'] & & v_6 \arrow[dr, "\beta"] & & w_6 \arrow[dl, "\delta"'] & & v_8 \arrow[dr, "\beta"] & & w_8 \arrow[dl, "\delta"'] & \cdots\cdots \\
& & w_1 & & & & w_3 & & & & w_5 & & & & w_7 & &
\end{tikzcd}$$

$$\phi_P(M(Z)) = \left\{ m \in M(Z) \;\middle|\; \exists m_2, m_3, m_4 : \; P \begin{pmatrix} m \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = 0 \right\}$$

**Proposition:** Let U be a K-subspace of an A-module M. Then the following statements are equivalent:

1. There exists a pointed finitely presented module (N, n) such that $U = H_{N,n}(M)$.

2. There exists a matrix P such that $U = \phi_P(M)$.

**Idea of proof:**

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \rightsquigarrow P \leftarrow \left\{\begin{array}{c} A^m \xrightarrow{P} A^n \xrightarrow{\pi} N \to 0 \\ \pi((1,0,\ldots,0)) =: n \end{array}\right.$$

# Pure-injective modules

**Definition:** Let M be an A-module and let L be an A-submodule. Then L is called a pure submodule if

$$\phi_P(L) = \phi_P(M) \cap L$$

for all matrices P.

A monomorphism $f : L \to M$ is called pure if $\mathrm{im}(f) \subseteq M$ is a pure submodule.

**Examples:** 1. Any direct summand is a pure submodule.

2. The canonical monomorphism $M \to M^{**}$ is pure.

**Definition:** A non-zero A-module N is called pure-injective if every pure monomorphism $f: N \longrightarrow M$ splits.

**Examples:** 1. A module is pure-injective iff it is a direct summand of a dual module.

$\forall M \in A\text{-Mod}$, $M \xrightarrow{ev} M^{**}$ is pure
If M pure-injective, then ev splits $\Rightarrow$ M is a $\oplus$-summand of $M^{**}$

2. Every finite-dimensional module is pure-injective.

$M \hookrightarrow M^{**}$ and $\dim_k M = \dim_k M^{**} \Rightarrow M \cong M^{**}$

3. Every injective module is pure-injective.

# The Ziegler Spectrum

<u>Definition:</u> An A-module M is **indecomposable** if, whenever $M \cong N \oplus L$, we have that $N = 0$ or $L=0$.

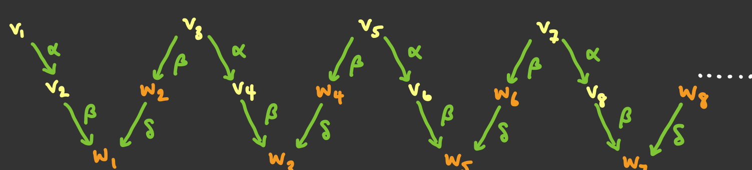
The **points of the Ziegler Spectrum** of A are the isomorphism classes of indecomposable pure-injective modules.

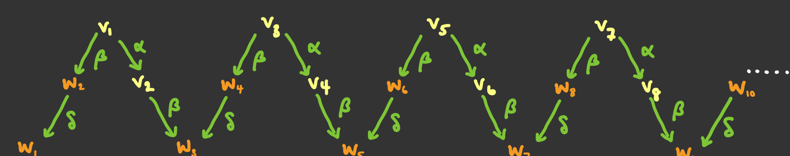
Denote the set of points by $Zg_A$.

**Examples of points of the Ziegler spectrum of K[X]:** For any k in K, the k-Prufer and the k-adic modules:

$$K^{(\mathbb{N})} \circlearrowright \begin{bmatrix} k & 1 & 0 & 0 & 0 & \cdots \\ 0 & k & 1 & 0 & 0 & \cdots \\ 0 & 0 & k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad K^{\mathbb{N}} \circlearrowright \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & k & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & k & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & k & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**Examples of points of the Ziegler spectrum of $\Lambda$:** The modules M(z) and M(y*)*:



**Definition:** A pair of matrices (P, Q) is called a **pp pair** if $\phi_P(M) \subseteq \phi_Q(M)$ for all A-modules M.

**Theorem (Ziegler, 1984):** Consider the following subsets of $Zg_A$ indexed by pp pairs (P, Q):

$$\left( \phi_P / \phi_Q \right) = \{ M \in Zg_A \mid \phi_P(M) \neq \phi_Q(M) \}$$

Then $Zg_A$ is a topological space with open sets:

$$\bigcup_{(P,Q) \in \Phi} \left( \phi_P / \phi_Q \right)$$

set of pp pairs

This space is called the **Ziegler spectrum of A**.

**Definition:** A topological space Z is called **quasi-compact** if whenever

$$Z = \bigcup_{i \in I} U_i$$

for U open sets, there exists a finite subset $F \subseteq L$ such that

$$Z = \bigcup_{i \in F} U_i$$

Ziegler showed that the open sets $(\phi_P / \phi_Q)$ are quasi-compact.

It follows that $Zg_A$ is quasi-compact:

Let $P = 0$ and $Q = 1$. Then $\phi_P(M) = M$ and $\phi_Q(M) = 0$. So $Zg_A = (\phi_P / \phi_Q)$

# Example: The Ziegler spectrum of K[T]

- the finite-dimensional module for each k in K and each m > 0 $\quad K[X]/(X-k)^m$

- the k-Prufer module for each k in K $\quad K^{(\mathbb{N})} \circlearrowleft \begin{bmatrix} k & 1 & 0 & 0 & 0 & \cdots \\ 0 & k & 1 & 0 & 0 & \cdots \\ 0 & 0 & k & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

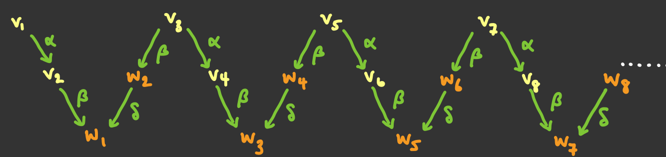
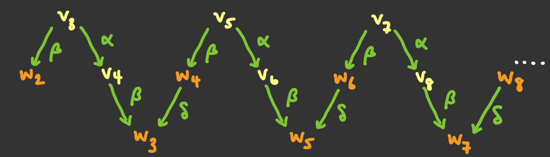
- the k-adic module for each k in K $\quad K^{\mathbb{N}} \circlearrowleft \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & k & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & k & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & k & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

- the field of fractions $\quad K(X)$

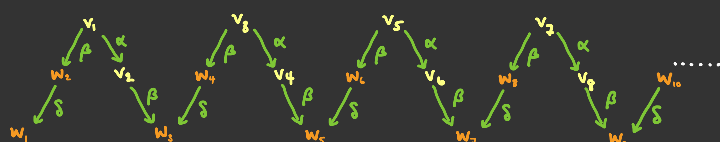
# Example: The Ziegler spectrum of $\Lambda$

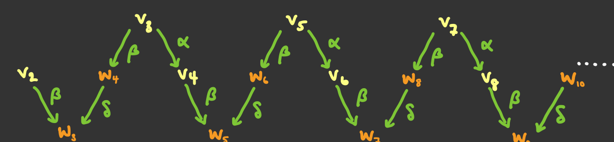
-the finite-dimensional $\Lambda$-modules

-the modules M(z) and M(z’) where

$z =$  $z' =$ 

-the dual modules M(y*)* and M(y’*)* with actions





-a module Ba(N) for every infinite-dimensional N in $Zg_{K[T]}$

# Finite representation-type and the Ziegler spectrum

Let $U$ denote the set of finite-dimensional points in $Zg_A$ where $A$ is a finite-dimensional algebra.

<u>Proposition:</u> 1. The closure $\bar{U}$ of $U$ is equal to $Zg_A$ (dense)

2. For every $M$ in $U$, the set $\{M\}$ is an open set (isolated)

3. For every $M$ in $U$, the set $\{M\}$ is a closed set (closed)

**Theorem:** Let A be a finite-dimensional algebra. Then the following statements are equivalent:

1. The set U is finite i.e. A has finite representation type.

2. The Ziegler spectrum has only finitely many points.

3. The Ziegler spectrum does not contain any infinite-dimensional points.

Proof: (2) $\Rightarrow$ (1): ✓

(1) $\Rightarrow$ (2), (3): $\mathcal{U} = \bigcup_{M \in \mathcal{U}} \{M\}$ is closed by Prop (3)

$\Rightarrow \mathcal{U} = \overline{\mathcal{U}} = Zg_A$ by Prop (1)

We have shown $(2) \Leftrightarrow (1) \Rightarrow (3)$, so remains to show that $(3) \Rightarrow (1)$.

$$(3) \Rightarrow \quad \mathrm{Zg}_A = U = \bigcup_{M \in U} \{M\}$$

By Prop(2), this is an open cover of $\mathrm{Zg}_A$ with no proper subcover

$\Rightarrow$ $U$ finite $\square$